arXiv:0903.0417v3 [math.DG] 12 Jun 2009

SEIBERG-WITTEN EQUATIONS AND RICCI CURVATURE ON 3-MANIFOLDS

CHANYOUNG SUNG

ABSTRACT. We prove an L^2-estimate involving Ricci curvature and a harmonic 1-form on a closed oriented Riemannian 3-manifold admitting a solution of any rescaled Seiberg-Witten equations. We also give a necessary condition to be a monopole class in some specific cases.

1. INTRODUCTION

A second cohomology class is called a *monopole class* if it arises as the first Chern class of a Spin^c structure for which the Seiberg-Witten equations admit a solution for every choice of a Riemannian metric. It is well-known by LeBrun [8, 9, 11] that the existence of a monopole class gives various curvature estimates of a Riemannian 4-manifold. These immediately give corresponding estimates [18] on 3-manifolds by using the dimensional reduction.

Theorem 1.1 ([18]). *Let (M,g) be a smooth closed oriented Riemannian 3-manifold with $b_1(M)\geq 1$. Suppose that it admits a solution of the Seiberg-Witten equations for a Spin^c structure $\mathfrak{s}$. Then*

$$\int_M (s_-)_g^2\ d\mu_g \geq \frac{16\pi^2|c_1(\mathfrak{s})\cup[\omega]|^2}{\int_M |\omega|_g^2\ d\mu_g},$$

where $(s_-)_g$ is $\min(s_g,0)$ at each point and s_g is the scalar curvature of g. Furthermore if the Seiberg-Witten invariant of $\mathfrak{s}$ is nonzero, then for a nonzero element ω in $H^1_{DR}(M)$

$$\int_M |r_g|^2 d\mu_g \geq \frac{8\pi^2|c_1(\mathfrak{s})\cup[\omega]|^2}{\int_M |\omega|_g^2\ d\mu_g},$$

where r_g is the Ricci curvature of g.

Date: June 12, 2009.
Key words and phrases. Seiberg-Witten equations, Ricci curvature, monopole class
MS Classification(2000): 57R57,57M50,53C99.
This work was supported by the Korea Research Foundation(KRF) grant funded by the Korea government(MEST). (No. 2009-0074404).

Here, the Seiberg-Witten invariant in case of $b_1(M) = 1$ means that of the chamber for arbitrarily small perturbations. We conjectured that the above Ricci curvature estimate still holds true when $c_1(\mathfrak{s})$ is a monopole class.

In this article, we show that it holds true if $c_1(\mathfrak{s})$ is a *strong monopole class* meaning that it is the first Chern class of a Spin^c structure of M which admits a solution of the rescaled Seiberg-Witten equations of $\mathfrak{s}$ for any rescaling and any Riemannian metric.

Following LeBrun [12], we define the f-rescaled Seiberg-Witten equations on a 3-manifold M to be

$$\begin{cases} D_A\Phi = 0 \\ F_A = (\Phi \otimes \Phi^* - \frac{|\Phi|^2}{2}\mathrm{Id})f, \end{cases}$$

where the rescaling factor f is a positive smooth function on M. An obvious but important fact is that if the Seiberg-Witten invariant of $\mathfrak{s}$ is nonzero, then $c_1(\mathfrak{s})$ is a strong monopole class. But it is not known yet whether every monopole class is a strong monopole class.

In general, it is very difficult to find a monopole class which has zero Seiberg-Witten invariant. In dimension 4, Bauer and Furuta [1, 2] devised a new refined invariant of Seiberg-Witten moduli space to prove the existence of a monopole class on some connected sums of Kähler surfaces. But it seems that no 3-dimensional example has been found yet. In the final section, we apply our curvature estimates to find a condition to be a monopole class in some specific examples.

For a brief introduction to the Seiberg-Witten theory, the readers are referred to [14, 19].

2. The Yamabe problem for Modified scalar curvature

Let (X, g) be a smooth closed oriented Riemannian 4-manifold and W_+ be the self-dual Weyl curvature. By the modified scalar curvature we mean

$$\mathfrak{S} \equiv s - \sqrt{6}|W_+|.$$

We will denote the set of $C^{2,\alpha}$ metrics for $\alpha \in (0,1)$ conformal to g by $[g]$. Assume that there exists a metric in $[g]$ with nonpositive $\int_X \mathfrak{S}\, d\mu$. Then as observed by Gursky [6] and LeBrun [10], the standard proof of the Yamabe problem [13] proves that there exists a $C^{2,\alpha}$ metric in $[g]$ such that $\mathfrak{S}$ is a nonpositive constant. As in Yamabe problem, it is a "minimizer" realizing

$$\mathfrak{Y}(X,[g]) \equiv \inf_{\tilde{g}\in[g]} \frac{\int_X \mathfrak{S}_{\tilde{g}}\, d\mu_{\tilde{g}}}{(\mathrm{Vol}_{\tilde{g}})^{\frac{1}{2}}}.$$

We also have

Lemma 2.1. *For $r \in [2, \infty]$,*

$$\mathfrak{Y}(X,[g]) = -\inf_{\tilde{g}\in[g]}\left(\int_X |\mathfrak{S}_{\tilde{g}}|^r d\mu_{\tilde{g}}\right)^{\frac{1}{r}} (Vol_{\tilde{g}})^{\frac{1}{2}-\frac{1}{r}}$$

where the infimum is realized only by the minimizer which is unique up to a constant multiplication.

Proof. We will use the technique of Besson, Courtois, and Gallot [4]. Let g be a minimizer. Let $\tilde{g} = u^2 g$, where $u : X \to \mathbb{R}^+$ is a C^2 function. Note that u satisfies the modified Yamabe equation

$$\mathfrak{S}_{\tilde{g}} u^3 = \mathfrak{S}_g u + 6\Delta_g u.$$

Therefore

$$\begin{aligned}
\left(\int_X |\mathfrak{S}_{\tilde{g}}|^r d\mu_{\tilde{g}}\right)^{\frac{1}{r}} (\mathrm{Vol}_{\tilde{g}})^{\frac{1}{2}-\frac{1}{r}} &= \left(\int_X |\mathfrak{S}_{\tilde{g}}|^r u^4 d\mu_g\right)^{\frac{1}{r}} \left(\int_X u^4 d\mu_g\right)^{\frac{1}{2}-\frac{1}{r}} \\
&\geq \frac{\int_X -\mathfrak{S}_{\tilde{g}} u^2 d\mu_g}{(\int_X d\mu_g)^{\frac{1}{2}}} \\
&= \frac{\int_X -(\mathfrak{S}_g + 6\frac{1}{u} d^* du)\ d\mu_g}{(\mathrm{Vol}_g)^{\frac{1}{2}}} \\
&= \frac{\int_X (-\mathfrak{S}_g + 6\frac{|du|^2}{u^2})\ d\mu_g}{(\mathrm{Vol}_g)^{\frac{1}{2}}} \\
&\geq \frac{\int_X -\mathfrak{S}_g\ d\mu_g}{(\mathrm{Vol}_g)^{\frac{1}{2}}},
\end{aligned}$$

where the first inequality is an application of the Hölder inequality, and the equality holds iff u is a positive constant. It also follows that any minimizer is a constant multiple of g. $\square$

3. Ricci curvature estimate

Let us start with the following lemma :

Lemma 3.1. *Let (M, g) be a smooth closed oriented Riemannian 3-manifold and $\mathfrak{s}$ be a Spin^c structure on it. If it admits a solution for a rescaled Seiberg-Witten equations, then any $C^{2,\alpha}$-metric $\tilde{g} \in [g]$ also has a solution of the rescaled Seiberg-Witten equations for $\mathfrak{s}$.*

Proof. We claim that if (A, Ψ) is a solution with respect to g, then $(A, e^{-\varphi}\Psi)$ is a solution with respect to $\tilde{g} = e^{2\varphi} g$. Mapping an orthonormal frame $\{e_1, e_2, e_3\}$ of g to an orthonormal frame $\{e^{-\varphi}e_1, e^{-\varphi}e_2, e^{-\varphi}e_3\}$ of $\tilde{g}$ gives a global isomorphism of two orthonormal frame bundles and hence a global

isometry of the Clifford bundles. Then the identity map between the spinor bundles is an isometry.

For a proof of the Spin^c Dirac equation, one is referred to [7], and the curvature equation is immediate from the fact that

$$|F_A|_{\tilde{g}} = e^{-2\varphi}|F_A|_g = e^{-2\varphi} f|\Psi|_g^2 = f|e^{-\varphi}\Psi|_{\tilde{g}}^2,$$

where f is the rescaling factor. □

Theorem 3.2. *Let (M, g) be a smooth closed oriented Riemannian 3-manifold with $b_1(M) \geq 1$ and $\mathfrak{s}$ be a Spin^c structure on it. Suppose that it admits a solution for the rescaled Seiberg-Witten equations for any rescaling.*

Then for any smooth metric $\tilde{g}$ conformal to g and any nonzero $\omega \in H^1_{DR}(M)$,

$$\int_M |r_{\tilde{g}}|^2 d\mu_{\tilde{g}} \geq \frac{8\pi^2|c_1(\mathfrak{s}) \cup [\omega]|^2}{\int_M |\omega|^2_{\tilde{g}}\ d\mu_{\tilde{g}}},$$

and the equality holds iff ω is $\tilde{g}$-harmonic, and $(M, \tilde{g})$ is a Riemannian submersion onto S^1 with totally geodesic fiber isometric to a compact oriented surface of genus ≥ 1 with a non-positive constant curvature metric whose volume form is a multiple of $*\omega$, and $[c_1(\mathfrak{s})]$ is a multiple of $[*\omega]$ in $H^2_{DR}(M)$, where $*$ denotes the Hodge star with respect to $\tilde{g}$.

Proof. In order prove the inequality, we may assume $c_1(\mathfrak{s}) \neq 0 \in H^2(M, \mathbb{R})$. Then the Seiberg-Witten equations have a irreducible solution for any metric in $[g]$, implying that there cannot exist a metric in $[g]$ with nonnegative scalar curvature, and hence there exists a smooth metric in $[g]$ with negative scalar curvature.

For notational convenience, let g be any smooth metric in $[g]$. Consider the product metric $g + dt^2$ on $M \times S^1$,where $t \in [0, 1]$ is a global coordinate of S^1. By the previous section, there exists a $C^{2,\alpha}$ metric $\hat{g} \in [g + dt^2]$ which minimizes $\mathfrak{Y}(M \times S^1, [g + dt^2])$ satisfying that $s - \sqrt{6}|W_+|$ is a negative constant.

Lemma 3.3. *$\hat{g}$ is invariant under the translation along S^1-direction.*

Proof. Let $\hat{g} = (g + dt^2)\psi$ for a positive smooth function ψ on $M \times S^1$, and we will show $\psi(x, t) = \psi(x, t + c)$ for $(x, t) \in M \times S^1$ for any c.

Since $(g + dt^2)\psi(x, t + c)$ is also a minimizer, by lemma 2.1 there exists a smooth positive function φ on S^1 such that

$$\psi(x, t + c) = \varphi(c)\psi(x, t)$$

for any (x,t). For any c,

$$\begin{aligned}\int_{M\times S^1}\psi(x,t)\ d\mu_{g+dt^2} &= \int_{M\times S^1}\psi(x,t+c)\ d\mu_{g+dt^2}\\ &= \varphi(c)\int_{M\times S^1}\psi(x,t)\ d\mu_{g+dt^2},\end{aligned}$$

where the first equality is due to the translation invariance of dt^2. Since $\psi>0$, we conclude that $\phi(c)=1$ for any c. □

We write the metric $\hat{g}$ as the warped form $h+f^2dt^2$ for $f:M\to\mathbb{R}^+$ where h is the metric f^2g on M. Let $\{e_1,e_2,e_3,e_4=\frac{\partial}{\partial t}\}$ be a local orthonormal frame $M\times S^1$ with respect to $h+dt^2$, and $\{\omega^i|i=1,\cdots,4\}$ its dual coframe. Recall the Cartan's first structure equations :

$$d\omega^i=-\omega^i_j\wedge\omega^j,$$

where ω^i_j are the connection 1-from of $h+dt^2$. Obviously ω^i_4 are all zero for all i. Take an orthonormal coframe of $\hat{g}$ as $\{\omega^1,\omega^2,\omega^3,f\omega^4\}$ and applying the Cartan's first structure equations to $\{\omega^1,\omega^2,\omega^3,f\omega^4\}$, we can see that the connection 1-forms $\hat{\omega}^i_j$ of $\hat{g}$ are given by

$$\hat{\omega}^i_j=\omega^i_j,\qquad \text{for } i,j=1,2,3$$

$$\hat{\omega}^4_j=\frac{\partial f}{\partial e_j}\omega^4\qquad \text{for } j=1,2,3.$$

Let (A,Φ) be a solution of the $\frac{1}{f}$-rescaled Seiberg-Witten equations for $\mathfrak{s}$ on (M,h), whose existence is guaranteed by lemma 3.1. Then it is a translation-invariant solution of the Seiberg-Witten equations for $\mathfrak{s}$ on $(M\times S^1,h+dt^2)$.

We claim that $(A,\frac{\Phi}{\sqrt{f}})$ is a solution of the unrescaled Seiberg-Witten equations for $\mathfrak{s}$ on $(M\times S^1,h+f^2dt^2)$. Let's denote the objects of $\hat{g}=h+f^2dt^2$

corresponding to that of $h+dt^2$ by $\hat{\cdot}$. The Spin^c Dirac equation reads

$$
\begin{aligned}
\hat{D}_A(f^{-\frac{1}{2}}\Phi) &= \sum_{i=1}^{4}\hat{e}_i\hat{\nabla}_{\hat{e}_i}(f^{-\frac{1}{2}}\Phi)\\
&= \sum_{i=1}^{4}\hat{e}_i(\frac{d}{d\hat{e}_i}(f^{-\frac{1}{2}}\Phi)+\frac{1}{2}(\sum_{j<k}\hat{\omega}_j^k(\hat{e}_i)\hat{e}_j\hat{e}_k+A(\hat{e}_i))f^{-\frac{1}{2}}\Phi)\\
&= \sum_{i=1}^{3}e_i(\frac{d}{de_i}(f^{-\frac{1}{2}}\Phi)+\frac{1}{2}(\sum_{j<k\leq 3}\omega_j^k(e_i)e_je_k+A(e_i))f^{-\frac{1}{2}}\Phi)\\
&\quad +\frac{\hat{e}_4}{2}(\sum_{j=1}^{3}\frac{\partial f}{\partial e_j}\omega^4(\hat{e}_4)e_j\hat{e}_4)f^{-\frac{1}{2}}\Phi\\
&= \sum_{i=1}^{3}e_i(-\frac{f^{-\frac{3}{2}}}{2}\frac{\partial f}{\partial e_i}\Phi+f^{-\frac{1}{2}}\frac{d\Phi}{de_i}+\frac{1}{2}(\sum_{j<k\leq 3}\omega_j^k(e_i)e_je_k+A(e_i))f^{-\frac{1}{2}}\Phi)\\
&\quad +\frac{1}{2}(\sum_{j=1}^{3}\frac{\partial f}{\partial e_j}\frac{1}{f}e_j)f^{-\frac{1}{2}}\Phi\\
&= f^{-\frac{1}{2}}D_A\Phi\\
&= 0,
\end{aligned}
$$

and the curvature equation reads

$$
\begin{aligned}
F_A^{\hat{+}} &= \frac{1}{2}(F_A+(*_hF_A)\wedge\hat{\omega}^4)\\
&\simeq \frac{1}{2}(F_A+(*_hF_A)\wedge\omega^4)\\
&= \frac{1}{f}(\Phi\otimes\Phi^*-\frac{|\Phi|^2}{2}\mathrm{Id})\\
&= (f^{-\frac{1}{2}}\Phi)\otimes(f^{-\frac{1}{2}}\Phi)^*-\frac{|f^{-\frac{1}{2}}\Phi|^2}{2}\mathrm{Id},
\end{aligned}
$$

where the equivalence in the second line means the identification as an endomorphism of the positive spinor bundle. (Mapping an orthonormal frame $\{e_1,\cdots,e_4\}$ of $h+dt^2$ to an orthonormal frame $\{e_1,\cdots,e_3,\hat{e}_4\}$ of $h+f^2dt^2$ gives a global isomorphism of two orthonormal frame bundles, inducing a global isometry of the Clifford bundles. Then the identity map between the spinor bundles is an isometry.)

Lemma 3.4.

$$\int_{M\times S^1}(\frac{2}{3}s_{g+dt^2}-2\sqrt{\frac{2}{3}}|W_+|_{g+dt^2})^2d\mu_{g+dt^2}\geq 32\pi^2((\pi^*c_1)^+)^2,$$

where $(\pi^*c_1)^+$ is the self-dual harmonic part of π^*c_1 with respect to $g+dt^2$, and $\pi : M\times S^1\to M$ is the projection map.

Proof. This immediately follows from LeBrun's method of theorem 2.2 in [10]. First by using lemma 2.1,

$$\int_{M\times S^1}(\frac{2}{3}s_{g+dt^2}-2\sqrt{\frac{2}{3}}|W_+|_{g+dt^2})^2d\mu_{g+dt^2} \quad\geq\quad \int_{M\times S^1}(\frac{2}{3}s_{\hat{g}}-2\sqrt{\frac{2}{3}}|W_+|_{\hat{g}})^2d\mu_{\hat{g}},$$

and the RHS is equal to

$$(\int_{M\times S^1}d\mu_{\hat{g}})^{\frac{1}{3}}(\int_{M\times S^1}|\frac{2}{3}s_{\hat{g}}-2\sqrt{\frac{2}{3}}|W_+|_{\hat{g}}|^3d\mu_{\hat{g}})^{\frac{2}{3}}, \tag{1}$$

because $\hat{g}$ has constat $\frac{2}{3}s-2\sqrt{\frac{2}{3}}|W_+|$. Now we use the fact that $(M\times S^1,\hat{g})$ admits a solution of the unrescaled Seiberg-Witten equations for $\mathfrak{s}$. Combining its Weitzenböck formula with the Weitzenböck formula for the self-dual harmonic 2-forms, we conclude that (1) is greater than or equal to $32\pi^2((\pi^*c_1)^+)^2$. □

Now using the above lemma, we get

$$\begin{aligned}
\int_M |r_g|^2\ d\mu_g &= \int_{M\times S^1}|r_{g+dt^2}|^2\ d\mu_{g+dt^2}\\
&= 8\int_{M\times S^1}(\frac{s^2_{g+dt^2}}{24}+\frac{1}{2}|W_+|^2_{g+dt^2})\ d\mu_{g+dt^2}\\
&\quad -8\pi^2(2\chi+3\tau)(M\times S^1)\\
&\geq \frac{1}{2}\int_{M\times S^1}(\frac{2}{3}s_{g+dt^2}-2\sqrt{\frac{2}{3}}|W_+|_{g+dt^2})^2d\mu_{g+dt^2}-0 \qquad (2)\\
&\geq 16\pi^2((\pi^*c_1)^+)^2 \qquad (3)\\
&\geq \frac{8\pi^2|c_1\cup[\omega]|^2}{\int_M|\omega|^2_g\ d\mu_g}, \qquad (4)
\end{aligned}$$

where the second equality is just the 4-dimensional Chern-Gauss-Bonnet theorem, and the first inequality is simple applications of Hölder inequality which was proved in LeBrun [11].

Lemma 3.5. *The equality holds iff ω is harmonic, and $(M\times S^1, g+dt^2)$ is a Kähler manifold of non-positive constant scalar curvature with the Kähler form a multiple of $*\omega+\omega\wedge dt$, and $[c_1(\mathfrak{s})]$ is a multiple of $[*\omega]$ in $H^2_{DR}(M)$.*

Proof. Let's first consider the case when $[c_1] \neq 0 \in H^2_{DR}(M)$. It is shown in [11] that both equalities in (2) and (3) hold iff $g + dt^2$ is a Kähler metric of negative constant scalar curvature with the Kähler form a multiple of $(\pi^* c_1)^+$. The equality in (4) holds iff

$$\omega = \omega^h = *c_1^h,$$

where $(\cdot)^h$ denotes the g-harmonic part.

When $[c_1] = 0 \in H^2_{DR}(M)$, the equality implies that the metric is Ricci-flat.(In fact, it's a flat manifold T^3/Γ, because the dimension is 3.) By the Weitzenböck formula for 1-forms, ω^h is a nonzero parallel 1-form. Then $*\omega^h + \omega^h \wedge dt$ is a nonzero parallel 2-form on $(M \times S^1, g + dt^2)$, and hence a Kähler form with the obvious complex structure compatible with the orientation. Conversely suppose that $(M \times S^1, g + dt^2)$ is scalar-flat Kähler. Since a Kähler curvature is a (symmetric) section of $\wedge^{1,1} \otimes \wedge^{1,1}$, on any scalar-flat Kähler surface the Riemann curvature restricted to self-dual two forms must be zero, and hence so is W_+. Then by the 4-dimensional Chern-Gauss-Bonnet theorem

$$\begin{aligned}\int_{M\times S^1} |r_{g+dt^2}|^2 d\mu_{g+dt^2} &= \int_{M\times S^1} (\frac{1}{3}(s_{g+dt^2})^2 + 4|W_+|^2_{g+dt^2}) d\mu_{g+dt^2} \\ &\quad - 8\pi^2(2\chi + 3\tau)(M \times S^1) \\ &= 0,\end{aligned}$$

giving the equality. □

Now if the equality holds, we have a parallel splitting of TM by ω and $*\omega$, each of which gives a manifold of constant scalar curvature of dimension 1 and 2 respectively. Thus the universal cover of (M, g) is isometric to $\mathbb{R}^3$ or $\mathbb{H}^2 \times \mathbb{R}^1$. Since M is a quotient by a discrete subgroup of $\mathbb{Z}^3$ or $PSL(2, \mathbb{R}) \times \mathbb{Z}$, (M, g) is obtained by identifying two boundaries of $\Sigma_g \times [0, 1]$ by an orientation preserving of (Σ, g_c) which is a compact Riemann surface of genus ≥ 1 with a constant curvature metric g_c. Then (M, g) is locally a Riemannian product of (Σ, g_c) and S^1, i.e. a Riemannian submersion onto S^1 with totally geodesic fiber (Σ, g_c).

Conversely, suppose that (M, g) is such an oriented Riemannian submersion $\pi : M \to S^1$. Then $(M \times S^1, g + dt^2)$ is a Kähler manifold with an obvious complex structure and a Kähler form $d\Omega + \pi^* ds \wedge dt$ where $d\Omega$ is the volume form of (Σ, g_c) and ds is the volume form of the base. Obviously the scalar curvature of g is just the constant scalar curvature of g_c and the first Chern class of Σ is a multiple of $d\Omega$ which is equal to $*\pi^* ds^2$.

This completes the proof. □

Remark As seen in the proof, the condition that (M, g) admits a solution

for the rescaled Seiberg-Witten equations for any rescaling is superfluous. It is enough to suppose that $(M \times S^1, \hat{g})$ has a solution for the unrescaled Seiberg-Witten equations. $\square$

Now let's discuss some immediate implications of the above theorem. First, by taking ω to be $*c_1^h(\mathfrak{s})$ where $(\cdot)^h$ denotes the harmonic part, we have

$$\int_M |r_{\tilde{g}}|^2 d\mu_{\tilde{g}} \geq 8\pi^2 \int_M |c_1^h(\mathfrak{s})|_{\tilde{g}}^2 \, d\mu_{\tilde{g}}.$$

More interestingly we can get a lower bound of L^2-norm of a harmonic 1-form on M :

Corollary 3.6. *Under the same hypothesis as theorem 3.2, if $\tilde{g}$ is not flat,*

$$(\int_M |\omega|_{\tilde{g}}^2 \, d\mu_{\tilde{g}})^{\frac{1}{2}} \geq \frac{2\sqrt{2}\pi|\alpha \cup [\omega]|}{(\int_M |r_{\tilde{g}}|^2 d\mu_{\tilde{g}})^{\frac{1}{2}}},$$

where α is a convex combination of any two such $c_1(\mathfrak{s})$'s.

4. Application

Our curvature estimates provide an easy toolkit in the study of a closed 3-manifold M with a non-torsion monopole. In [18], we derived the inevitability of collapsing when such a manifold has zero Yamabe invariant which implies the existence of a sequence of unit-volume Riemannian metrics $\{g_i\}$ on M satisfying $\inf_i \int_M s_{g_i}^2 d\mu_{g_i} = 0$. We also find a necessary condition to be a monopole class in a specific example as follows :

Proposition 4.1 ([18])**.** *Let M be a closed oriented 3-manifold which fibers over the circle with a periodic monodromy, and N be a closed oriented 3-manifold with $b_1(N) = 0$.*

Then the rational part of a monopole class of $M\#N$ is of the form $m[F]$ for an integer m satisfying $|m| \leq |\chi(F)|$, where $\chi(F)$ is the Euler characteristic of the fiber F.

(N.B. : In the statement of the theorem 1.4 of [18], $b_1(N) = 0$ is missing by mistake.)

We give a little generalization of this.

Proposition 4.2. *Let M_i for $i = 1, \cdots, n$ be a closed oriented 3-manifold which fibers over the circle with a periodic monodromy, and N be any closed oriented 3-manifold.*

Then the rational part of a monopole class of $M_1\#\cdots\#M_n\#N$ is of the form $\beta + \sum_{i=1}^n m_i[F_i]$ for $\beta \in H^2(N, \mathbb{Z})$ and an integer m_i satisfying $|m_i| \leq -\chi(F_i)$, where $\chi(F_i)$ is the Euler characteristic of the fiber F_i in M_i.

Proof. Let α be a monopole class of $X = M_1 \# \cdots \# M_n \# N$.

First if any F_i is a 2-sphere, then the only possibility for M_i is $S^1 \times S^2$. Letting $[\omega]$ be the Poincaré-dual of F_i, we only have to show that it pairs zero with α. Let $0 < \varepsilon \ll 1$.

Take a metric of positive scalar curvature on the M_i. For the connected sum, we take a small ball on M_i, and a representative ω of $[\omega]$ to be supported outside of that ball. Then perform the Gromov-Lawson type surgery [5, 16, 17] on it keeping the positivity of scalar curvature to get a compact manifold M_i' with a cylindrical boundary. And then contract it small enough so that

$$\int_{M_i'} |\omega|^2 d\mu \leq \varepsilon.$$

On the other part of X, we put any metric such that it satisfies

$$\int (s_-)^2 d\mu < 1, \tag{5}$$

and perform the Gromov-Lawson surgery such that the cylindrical boundary matches with that of the above-made M_i' while still satisfying (5).

After gluing these two pieces, we have that

$$\int_X |\omega|^2 d\mu \leq \varepsilon \qquad \text{and} \qquad \int_X (s_-)^2 d\mu < 1.$$

Applying theorem 1.1, we get

$$|4\pi\alpha \cup [\omega]|^2 < \varepsilon,$$

which proves $\alpha \cup [\omega] = 0$.

Secondly, let's consider the case of M_i with $\chi(F_i) \leq 0$. Let $[\omega] \in H^1(M_i, \mathbb{R})$. In this case, $H^1(M_i, \mathbb{R})$ is generated by

$$\pi_i^* dt, \qquad \text{and} \qquad \{[\sigma] \in H^1(F_i, \mathbb{R}) | f_i^*[\sigma] = [\sigma]\},$$

where $\pi_i : M_i \to S^1$ is the projection map and f_i is the monodromy diffeomorphism.

When $[\omega]$ is one of the latter ones, we have to show that it pairs zero with α. We can express ω as

$$\frac{1}{d_i} \sum_{n=1}^{d_i} (f_i^n)^* \sigma$$

for such σ satisfying $[f^*\sigma] = [\sigma]$, where d_i is the order of f_i.

By taking a f_i-invariant metric on F_i, we can put a locally-product metric on M_i such that π_i is a Riemannian submersion with totally geodesic fibers onto a circle of radius ε and

$$\int (s_-)^2 d\mu < \varepsilon, \tag{6}$$

We can take a small simply-connected open set B in F_i, which is invariant under f_i, and take a representative σ of the above $[\sigma] \in H^1(F_i, \mathbb{R})$ to be supported outside of B.

For the connected sum, we perform the Gromov-Lawson surgery on $B \times I(\frac{\varepsilon}{10}) \subset M_i$ where $I(\frac{\varepsilon}{10})$ is the interval of length $\frac{\varepsilon}{10}$ to get M_i' with a cylindrical end while still satisfying (6). On the other part of X, as before we put a metric with cylindrical end isometric to that of this M_i' while satisfying (5).

After gluing two pieces, we have

$$\int_X |\omega|^2 d\mu \leq C\varepsilon \qquad \text{and} \qquad \int_X (s_-)^2 d\mu < 1+\varepsilon,$$

for a constant $C > 0$. Hence by theorem 1.1

$$|4\pi\alpha \cup [\omega]|^2 < \varepsilon(1+\varepsilon),$$

proving $\alpha \cup [\omega] = 0$.

Finally when $\omega = \pi_i^* dt$, the adjunction inequality gives

$$|\langle \alpha, [F_i] \rangle| \leq -\chi(F_i).$$

□

Remark As noted, $M_i \times S^1$ for M_i as above admits a Kähler metric of constant scalar curvature, and each M_i admits a d_i-fold covering space which is $F_i \times S^1$. For M_i with $F_i = S^2$, one can use the argument of gluing moduli spaces of Seiberg-Witten equations along the cylindrical end to prove $4\pi\alpha \cup dt = 0$. □

It seems plausible to conjecture :

Conjecture 4.3. *Let N_i for $i = 1, \cdots, n$ be a closed oriented 3-manifold. Then (the rational part of) a monopole class of $N_1 \# \cdots \# N_n$ is expressed as $\sum_i^n \alpha_i$ where α_i is a monopole class of N_i.*

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National Institute for Mathematical Sciences, 628 Daeduk-Boulevard, Yuseong-gu, Daejeon, KOREA
E-mail address: cysung@nims.re.kr